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COMMENT

On the f–α spectrum of one-hump maps

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Abstract. Based on a perturbative procedure developed to solve for the universal parameters of one-hump maps, the corresponding fractal dimensions D_q and the f-α spectrum are studied.

Recently a lot of interest has been concentrated on fractal sets which are found to exist in a large variety of non-linear systems. In these systems, the long-time stochastic motion lies on a complicated manifold in phase space called the strange attractor. One-dimensional maps are found to provide a suitable framework modelling most of these systems. In this comment we consider maps of the form $x_{n+1} = 1 - \lambda|x_n|^z$, on the interval $(-1, 1)$, where λ is the control parameter and z is the degeneracy of the critical point. The period-doubling cascade of such maps accumulates at λ_∞, where the map possesses a 2^∞ orbit. The associated universal behaviour is characterised by the function $g(x)$ which satisfies the functional equation (Feigenbaum 1978, 1979)

$$g(x) = -\alpha_{\rm pd} g(g(x/\alpha_{\rm pd}) \tag{1}$$

where $\alpha_{\rm pd}$ is the universal scaling factor defined for the separation between two adjacent fixed points in the period-doubling scenario (Feigenbaum 1980). The iterates of $g(x)$ form a nearly self-similar Cantor set, called the Feigenbaum attractor. The nature of the self-similarity of such an attractor as well as analytic approximations for the first three dimensions D_0, D_1 and D_2 have been recently studied by Hu (1987).

This comment is meant to supplement these investigations, but our method is significantly different in that an analytic method of solving (1) is used to get $g(x)$. Using this, we derive equations to be solved for the generalised dimensions D_q, for any value of z. These are used to compute the spectrum of α values and their densities $f(\alpha)$ (Benzi *et al* 1984, Halsey *et al* 1986), which provide a complete characterisation of the scaling structure of the attractor.

A perturbative scheme was developed recently to evaluate the universal parameters of one-hump maps (Singh 1985, Ambika and Babu Joseph 1986, 1988). The function $g(x)$ is first written as a power series in x^z as

$$g(x) = 1 + \sum_{n=1}^{\infty} P_n|x|^{nz}. \tag{2a}$$

In the neighbourhood of $x=0$, $g(x)$ is positive for any z and so $g(g(x))$ can also be expanded into a similar power series:

$$g(g(x)) = 1 + \sum_{r=1}^{\infty} P_r + \left(P_1 \sum_{r=1}^{\infty} r\,z\,P_r\right)|x|^z + \left(P_2 \sum_{r=1}^{\infty} r\,z\,P_r + P_1^2 \sum_{r=1}^{\infty} \frac{r\,z(rz-1)}{2!} P_r\right)|x|^{2z} + \dots . \tag{2b}$$

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The perturbative method depends on a proper redefinition of the coefficients P_n, which brings about a reshuffling of the terms leading to accelerated convergence. Thus we define

$$P_n \alpha_{\rm pd}^n = S_n |\alpha_{\rm pd}|^z. \tag{3}$$

Using (2a), (2b) and (3) in (1) and equating coefficients of $|x|^{nz}$ on both sides, we get

$$\frac{1}{\alpha_{\rm pd}} + 1 + |\alpha_{\rm pd}|^z \sum_{r=1}^{\infty} \frac{S_r}{\alpha_{\rm pd}^r} = 0 \qquad n = 0 \tag{4}$$

$$\frac{1}{z} + \sum_{r=1}^{\infty} \frac{rS_r}{\alpha_{\rm pd}^{r-1}} = 0 \qquad n = 1 \tag{5}$$

$$S_n\left(1 - \frac{1}{|\alpha_{\rm pd}|^{z(n-1)}}\right) + \sum_{l\geq 2}^{n} \sum_{r\geq 1}^{\infty} \binom{rz}{l} \frac{S_r}{\alpha_{\rm pd}^{r-1}}$$

$$\times \sum_{m_1\geq 1,\, m_2\geq 1 \ldots m\geq 1} \frac{S_{m_1} S_{m_2} \ldots S_{m_l}}{|\alpha_{\rm pd}|^{z(n-l)}} \delta_{m_1+m_2+\ldots+m_l, n}$$

$$n = 2, 3, 4, \ldots . \tag{6}$$

To solve these coupled non-linear equations, S_n are expanded into inverse powers in $\alpha_{\rm pd}$ as

$$S_n = \sum_{m=0}^{\infty} \frac{S_{nm}}{\alpha_{\rm pd}^m}. \tag{7}$$

Using (7) in (5) and (6) a set of equations result and these can be solved successively for S_{nm}. Then $g(x)$ is given as

$$g(x) = 1 + \sum_{n=1}^{\infty} \left(|\alpha_{\rm pd}|^{z-n} \sum_{m=0}^{\infty} \frac{S_{nm}}{\alpha_{\rm pd}^m}\right)|x|^{nz} \tag{8}$$

and the scale factor $\alpha_{\rm pd}$ is given by (4)

$$\frac{1}{\alpha_{\rm pd}} + 1 + |\alpha_{\rm pd}|^z \sum_{r=1}^{\infty} \sum_{m=0}^{\infty} \frac{S_{rm}}{\alpha_{\rm pd}^{r+m}} = 0. \tag{9}$$

For any z value, the first few coefficients S_{nm} work out to be

$$S_{10} = -\frac{1}{z} \qquad S_{11} = \frac{-(z-l)}{z^2} \qquad S_{20} = \frac{(z-l)}{2z^2} \qquad \ldots . \tag{10}$$

Then equation (9) is

$$\frac{1}{\alpha_{\rm pd}} + 1 = |\alpha_{\rm pd}|^z \left(\frac{1}{z\alpha_{\rm pd}} + \frac{(z-1)}{2z^2\alpha_{\rm pd}^2} + \ldots\right). \tag{11}$$

The series in the brackets is found to be asymptotic in nature and can be replaced by its Padé approximant (Ambika and Valsamma 1988). Here considering the lowest approximation, we use the [1/1] approximant to get

$$\frac{1}{\alpha_{\rm pd}} + 1 = |\alpha_{\rm pd}|^z \left[z\alpha_{\rm pd}\left(1 - \frac{(z-1)}{2z\alpha_{\rm pd}}\right)\right]^{-1}. \tag{12}$$

Using the coefficients in (10), (8) gives

$$g(x)=1-|\alpha_{\rm pd}|^{z}\left(\frac{1}{z\alpha_{\rm pd}}+\frac{(z-1)}{z^{2}\alpha_{\rm pd}^{2}}\right)|x|^{z}+\frac{|\alpha_{\rm pd}|^{z}(z-1)}{2z^{2}\alpha_{\rm pd}^{2}}|x|^{2z}+\ldots. \tag{13}$$

The attractor of one-hump maps consists of a set of points $(x_i)^k$ generated by the map

$$x_{i+1}=g(x_i) \tag{14}$$

starting from $x_0=0$. The whole set lies in the interval $(-1/\alpha_{\rm pd},1)$ but can be rescaled (Hu 1987) such that the new set $(x_i')^k$ lies in the interval (0, 1). The attractor can be divided into subsets $s_1, s_2, s_3, \ldots$, each having the same structure but rescaled by different factors $1/s_j$ where

$$s_j=\frac{1}{x'_{2j}-x'_{2j-1}}. \tag{15}$$

The iterates after rescaling and re-ordering are

$$x_1'=0 \qquad x_2'=\frac{1}{\alpha_{\rm pd}}$$

$$x_3'=\frac{1}{(1+1/\alpha_{\rm pd})}\left[g\left(\frac{1}{\alpha_{\rm pd}}\right)+\frac{1}{\alpha_{\rm pd}}\right] \qquad x_4'=1.$$

The corresponding subsets are

$$s_1=\alpha \qquad s_2=\frac{(1+1/\alpha_{\rm pd})}{[1-g(1/\alpha_{\rm pd})]}. \tag{16}$$

The generalised dimensions D_q can be computed by defining a partition function Γ as (Halsey and Jensen 1986, Grassberger 1985)

$$\Gamma=\sum_{i=1}^{N}\frac{p_i{}^{q}}{l_i{}^{\tau}}. \tag{17}$$

For large N, Γ is of order unity and τ is related to the generalised dimensions through the relation

$$\tau=(q-1)D_q. \tag{18}$$

Here $l_i=1/s_i$ and all the p are equal and hence when we consider R subsets of the attractor, (17) gives

$$\sum_{j=1}^{R}(s_j)^{\tau}=R^{q}. \tag{19}$$

This leads to a transcendental equation for τ and D_q is then computed using (18) for each q.

The singularities α are defined as (Halsey *et al* 1986)

$$\alpha=\frac{\partial\tau}{\partial q} \tag{20}$$

and the corresponding fractal dimension $f(\alpha)$, obtained as

$$f(\alpha)=\alpha q-\tau. \tag{21}$$

For the two subsets considered in (16), (19) gives

$$\alpha_{\mathrm{pd}}^{\tau}+\alpha_{\mathrm{pd}}^{z\tau}\left(1+\frac{(z-1)}{2z\alpha_{\mathrm{pd}}}\right)^{-\tau}=2^{q}. \tag{22}$$

This equation is to be solved to get τ and hence D_q for any q except $q=1$. For this special limiting case we use the expressions derived by Hu (1987) and get

$$D_1=2\ln 2\left[(z+1)\ln\alpha_{\mathrm{pd}}-\ln\left(1+\frac{(z-1)}{2z\alpha_{\mathrm{pd}}}\right)\right]^{-1}. \tag{23}$$

For a range of z values from 1.2 to 10, we compute α_{pd} using (12). The dimensions D_q calculated using (22) and (23) are shown in figure 1 for a few typical q values. We find that for any q, D_q varies continuously with z. Moreover as z increases the D_q get spread out over a wide range and the spread is asymmetrical about D_0. However, for large q values, the D_q values get closer and closer. In particular D_0 increases with z and reaches almost saturation for large z. For positive q values, D_q increases first, reaches a maximum and then decreases, while for negative q values, D_q is found to increase monotonically with z.

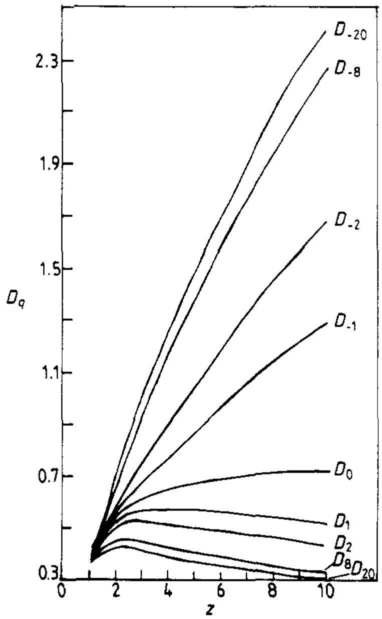

Figure 1. Dependence of the fractal dimensions D_q on the degeneracy of the critical point of the map.

Using (20) and (21), we calculate α and $f(\alpha)$ and the f against α curve for a few specific z values are shown in figure 2. One interesting observation is that the f and α values crowd near the $D_{\pm\infty}$ regions for large z, while they are distributed uniformly over the curve for small z values. Moreover, the values near the $D_{+\infty}$ region are not much different for different z while they differ considerably near the $D_{-\infty}$ region. This points to the fact that the difference in the corresponding attractors lies mainly around their most rarefied regions.

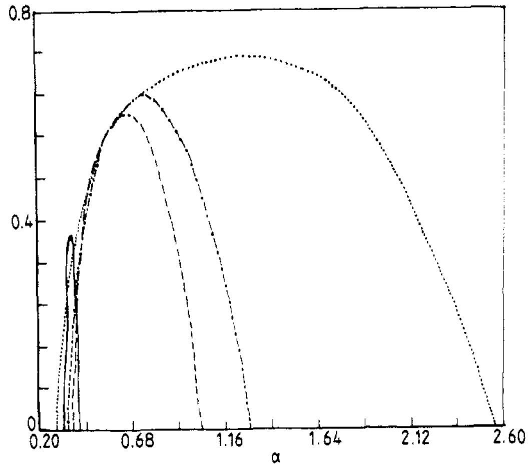

Figure 2. The f against α curve for a few typical z values: $(\cdots\cdot)$ $z=10$, $(-\cdot-)$ $z=4$, $(---)$ $z=3$ and $(——)$ $z=1.2$.

For quadratic maps, we have

$$D_{-\infty}=\frac{\ln 2}{\ln \alpha_{\rm pd}}$$

and

$$D_{+\infty}=\frac{\ln 2}{\ln \alpha^2_{\rm pd}}$$

(Halsey *et al* 1986). In general we may define $D_{+\infty}$ for arbitrary z as $D_{+\infty}=\ln 2/\ln \alpha^z_{\rm pd}$. The values of $D_{\pm\infty}$ taken from the endpoints of the f-α curves and those calculated using the above expressions are given in table 1. We find the agreement is remarkable for $D_{-\infty}$ while it is not so good for $D_{+\infty}$. This is because for large positive q values, (22) could be solved only with less accuracy. We note that $D_{+\infty}$ and $D_{-\infty}$ values show the limiting behaviour to be expected from figure 1.

In conclusion, we mention that our method is mainly analytical in nature, even though some computations are unavoidable in the end. We could extend the earlier work on the fractal dimensions of one-hump maps by providing all the relevant D_q values for a range of z values. The f-α spectrum has been drawn for a few typical z

Table 1. Calculated values and values from graph for $D_{\pm\infty}$ for the z values shown in figure 2.

	$D_{+\infty}$		$D_{-\infty}$	
z	By calculation	From graph	By calculation	From graph
1.2	0.342 880	0.341	0.411 460	0.4187
3	0.348 578	0.360	1.045 640	1.040
4	0.325 018	0.3467	1.300 072	1.2933
10	0.254 714	0.267	2.547 140	2.547

values. We note that, even though the f-α spectrum for $z=2$ has been studied numerically (Halsey and Jensen 1986), that for other z values has not been discussed so far.

Our aim in this comment has been to indicate the main trend of our method and so we work with the crudest approximation. But even then, the agreement with available numerical values is quite good. The method has the advantage that it is straightforward to improve the results by including more terms in the series in (12) and (13) and considering higher-order Padé approximants (Ambika and Valsamma 1988). It is also possible to consider more subsets of the attractor in (19) leading to greater accuracy.

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